Integration Workshop 2005
Project on the Stone-Weierstrass Theorem

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The goal of this project is to prove the Stone-Weierstrass Theorem. In 1885 Weierstrass proved that on a closed interval every polynomial can be uniformly approximated arbitrarily closely by polynomials. This result was generalized in 1937 by Stone.

Suppose that we have a collection of continuous functions on a compact Hausdorff space which is closed under addition, multiplication and scalar multiplication. Also suppose that for any two distinct points in the space, there is a continuous function in the collection which takes distinct values at these points. Then the theorem says that this collection approximates any continuous function arbitrarily closely.

Let $X$ be a compact Hausdorff topological space. Let $C(X, \mathbb{R})$ (respectively, $C(X, \mathbb{C})$) denote the set of all continuous real-valued (respectively, complex-valued) functions on $X$. We provide $C(X, \mathbb{R})$ and $C(X, \mathbb{C})$ with sup-norm metric. That is, for $f, g \in C(X, \mathbb{R})$ or $C(X, \mathbb{C})$, $d(f, g) = \|f - g\|_u = \sup_{x \in X} |f(x) - g(x)|$.

Let $\mathcal{A}$ be a subset of $C(X, \mathbb{R})$ (respectively, $C(X, \mathbb{C})$). $\mathcal{A}$ separates points if for every $x, y \in X$, $x \neq y$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. $\mathcal{A}$ is an subalgebra if $\mathcal{A}$ is a real (respectively, complex) vector subspace of $C(X, \mathbb{R})$ (respectively, $C(X, \mathbb{C})$) and $fg \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$. $\mathcal{A} \subset C(X, \mathbb{R})$ is called a lattice if max($f, g$) and min($f, g$) are in $C(X, \mathbb{R})$ whenever $f$ and $g$ are.

Exercise: Show that $C(X, \mathbb{R})$ (and therefore $C(X, \mathbb{C})$) separates points.

Hint: Use the fact that a compact Hausdorff space is normal, hence Urysohn lemma holds.

Example: Let $X = \{x_1, ..., x_n\}$ with the discrete topology. Consider $h : C(X, \mathbb{R}) \to \mathbb{R}^n$ defined by $h(f) = (f(x_1), ..., f(x_n))$. Show that $h$ is an algebra isomorphism if the multiplication in $\mathbb{R}^n$ is defined coordinate-wise.

The Stone-Weierstrass Theorem. Let $X$ be a compact Hausdorff topological space. If $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbb{R})$ which separates points, then either $\mathcal{A} = C(X, \mathbb{R})$ or $\mathcal{A} = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in X$. The first alternative is the case exactly when $\mathcal{A}$ contains all the constant functions in $C(X, \mathbb{R})$.

We first prove several lemmas. The first lemma is the special case of the theorem for $X = \{x_1, x_2\}$.

Lemma. The only subalgebras of $\mathbb{R}^2$ are $\mathbb{R}^2$, $\{(0,0)\}$, $\{(r,0) : r \in \mathbb{R}\}$, $\{(0,r) : r \in \mathbb{R}\}$, $\{(r,r) : r \in \mathbb{R}\}$.

Hint: If a subalgebra $\mathcal{A}$ of $\mathbb{R}^2$ which contains $(a,b) \in \mathbb{R}^2$ with $a \neq b$, $a \neq 0$ and $b \neq 0$, then $(a^2, b^2)$ is also in $\mathcal{A}$. Conclude that $\mathcal{A} = \mathbb{R}^2$ in this case. Determine what happens in the cases where there is no such element in $\mathcal{A}$.

We have to do a little calculus in preparation for the next lemma.

Lemma. The Taylor’s series of $f(t) = (1 - t)^{1/2}$ at 0 converges absolutely and uniformly to $f(t)$ on $[-1, 1]$.

Proof: Step 1: Show that the Taylor’s series of $f(t)$ converges absolutely and uniformly on $[-1, 1]$. Hint: Use the Stirling’s formula:

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n^{n+1/2}} e^{-n}} = 1.$$
Step 2: Let \( g(t) \) be the limit of the Taylor’s series for \( t \in [-1, 1] \). Show that \( 2(1-t)g'(t) = -g(t) \) for \( t \in (-1, 1) \). Solve this differential equation to conclude that \( g(t) = f(t) \) for \( t \in (-1, 1) \).

Step 3: Note that both \( f \) and \( g \) are continuous to finish the proofs.

One of the consequences of Stone-Weierstrass Theorem will be that the polynomials are dense in \( C([-1, 1], \mathbb{R}) \) in the uniform norm. In the next lemma we prove that on \([-1, 1], |x| \) is a limit of a sequence of polynomials which vanish at 0.

**Lemma.** For any \( \epsilon > 0 \) there exists a polynomial with real coefficients such that \( P(0) = 0 \) and \( |x| - P(x)| < \epsilon \) for all \( x \in \mathbb{R} \).

**Proof:** Step 1: Use the previous lemma to choose a polynomial \( Q(x) \) such that \( |(1-t)^{1/2} - Q(t)| < \epsilon/2 \) for \( t \in [-1, 1] \).

Step 2: Let \( t = 1 - x^2 \) and \( R(x) = Q(1 - x^2) \) to get a polynomial \( R(x) \) satisfying \( |x - R(x)| < \epsilon/2 \) for \( x \in [-1, 1] \).

Step 3: Finally, use \( R(x) \) to construct a polynomial \( P(x) \) such that \( |x - P(x)| < \epsilon \) for \( x \in [-1, 1] \) and \( P(0) = 0 \).

Now we prove that every closed subalgebra is a lattice:

**Lemma.** If \( \mathcal{A} \) is a closed subalgebra of \( C(X, \mathbb{R}) \), then \( |f| \in C(X, \mathbb{R}) \) for every \( f \in C(X, \mathbb{R}) \), and \( \mathcal{A} \) is a lattice.

**Proof:** Step 1: Let \( \epsilon > 0 \). For \( 0 \neq f \in C(X, \mathbb{R}) \) let \( h = f/\|f\|_u \), and use the previous lemma to obtain \( \|h - P \circ h\|_u < \epsilon \).

Step 2: Observe that \( P \circ h \in \mathcal{A} \).

Step 3: Since \( \mathcal{A} \) is closed and \( \epsilon > 0 \) is arbitrary, conclude that \( |f| \in \mathcal{A} \). This finishes the proof of the first claim.

Step 4: Discover a way of expressing \( \max(f, g) \) and \( \min(f, g) \) in terms of \( f \) and \( g \) using the algebra operations and \(| \cdot |\). Use this to show that \( \mathcal{A} \) is a lattice.

The last lemma says that if a closed lattice is sufficiently large, then it is quite large.

**Lemma.** Let \( \mathcal{A} \) be a closed lattice of \( C(X, \mathbb{R}) \). If \( f \in C(X, \mathbb{R}) \)and for every \( x, y \in X \) there exists \( g_{xy} \in \mathcal{A} \) such that \( g_{xy}(x) = f(x) \) and \( g_{xy}(y) = f(y) \), then \( f \in \mathcal{A} \).

**Proof:** Step 1: Let \( \epsilon > 0 \). For each pair \( x, y \in X \) let \( U_{xy} = \{z \in X : f(z) < g_{xy}(z) + \epsilon\} \) and \( V_{xy} = \{z \in X : f(z) > g_{xy}(z) - \epsilon\} \). Show that these sets open and contain \( x \) and \( y \).

Step 2: Fix \( y \). As \( x \) ranges over \( X \), the sets \( U_{xy} \) cover \( X \). Use compactness to find a finite subcover corresponding to a finite set of points, say, \( x_1, \ldots, x_n \in X \). Let \( g_y = \max(g_{x_1y}, g_{x_2y}, \ldots, g_{xn y}) \). Then \( f < g_y + \epsilon \) on \( X \) and \( f > g_y - \epsilon \) on the open set \( V_y = \bigcap_{1 \leq i \leq n} V_{x_i y} \) which contains \( y \).

Step 3: Now as \( y \) ranges over \( X \), the sets \( V_y \) form an open cover of \( X \). Get a finite subcover corresponding to the points, say, \( y_1, y_2, \ldots, y_m \in X \) and let \( g = \min(g_{y_1}, g_{y_2}, \ldots, g_{y_m}) \). Then \( \|f - g\|_u < \epsilon \).

Step 4: Use the fact that \( \mathcal{A} \) is a closed lattice to finish the proof.

We are now ready to prove the Stone-Weierstrass Theorem.
Proof of The Stone-Weierstrass Theorem: Step 1: For any pair of distinct points \( x, y \in X \), let \( A_{xy} = \{(f(x), f(y)) : f \in A\} \subset \mathbb{R}^2 \). Observe that \( A_{xy} \) is a subalgebra of \( \mathbb{R}^2 \).

Step 2: Use The last two lemmas to conclude that \( A = C(X, \mathbb{R}) \) if \( A_{xy} = \mathbb{R}^2 \) for all \( x, y \).

Step 3: If not, then there exist \( x, y \) such that \( A_{xy} \) is a proper subalgebra of \( \mathbb{R}^2 \). Use the first lemma to decide which subalgebra it can be. Conclude that there exists \( x_0 \in X \) such that \( f(x_0) = 0 \) for all \( f \in A \).

Step 4: Show that \( x_0 \) is unique since \( A \) separates points.

Step 5: Use The last two lemmas again to conclude that \( A = \{ f \in C(X, \mathbb{R}) : f(x_0) = 0 \} \).

Step 6: Finally observe that this can not be the case if \( A \) contains the constant functions. □

Corollary. Let \( X \) be a compact subset of \( \mathbb{R}^n \). Then the set of all polynomials is dense in \( C(X, \mathbb{R}) \).

Remark: We want to prove a complex version of Stone-Weierstrass Theorem. But this would not be true without a further assumption: Consider the unit circle \( X = \{ z \in \mathbb{C} : |z| = 1 \} \) in the complex plane \( \mathbb{C} \). Then the polynomials in \( z \) with complex coefficients will separate points in \( X \), but they will not be dense in \( C(X, \mathbb{C}) \). For instance, \( f(z) = \bar{z} \) is not a limit of polynomials.

Exercise: Show that for \( 0 < \epsilon < 1 \) there is no polynomial \( P(z) \) such that \( |\bar{z} - P(z)| < \epsilon \) for all \( |z| = 1 \).

Hint: Show that \( \int_X zP(z)dz = 0 \). Then compute \( \int_X |z|^2dz \) using \( |z|^2 = z(\bar{z} - P(z)) + zP(z) \) to obtain a contradiction.

The Complex Stone-Weierstrass Theorem. Let \( X \) be a compact Hausdorff topological space. If \( A \) is a closed subalgebra of \( C(X, \mathbb{C}) \) which separates points and is closed under complex conjugation, then either \( A = C(X, \mathbb{C}) \) or \( A = \{ f \in C(X, \mathbb{C}) : f(x_0) = 0 \} \) for some \( x_0 \in X \). The first alternative is the case exactly when \( A \) contains all the constant functions in \( C(X, \mathbb{C}) \).

Hint: Apply the Stone-Weierstrass Theorem to the subalgebra \( A_{\mathbb{R}} \) of \( C(X, \mathbb{R}) \) consisting of all \( (f + \bar{f})/2 \) and \( (f - \bar{f})/(2i) \) for \( f \in A \).